# Thermostatistics of the g-on gas

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**Abstract.** In this study, the particles of the quantum gases, namely bosons and fermions are called $g$-ons by using the parameter of the fractional exclusion statistics $g$, where $0 \leq g \leq 1$. With this point of departure, the distribution function of the $g$-on gas is obtained by the variational, steepest descent and statistical methods. The distribution functions which are found by means of these three methods are compared. It is shown that the thermostatistical formulations of quantum gases can be unified. By suitable choices of $g$, standard relations of statistical mechanics of the Bose and Fermi systems are recovered.

**PACS.** 05.20.-y Classical statistical mechanics – 03.65.-w Quantum mechanics

## 1 Introduction

Particles of quantum systems, including well known fermions and bosons could be named as $g$-ons which obey fractional exclusion statistics. The aim of this study is to develop the distribution function of a $g$-on gas by means of variational, steepest descent and statistical methods and discuss the result. Furthermore, another purpose is to unify the different thermostatistical formulations of quantum gases where the concomitant distribution function of $g$-on gas is used.

The outline of this paper is as follows: in Section 2.1, Haldane statistics is obtained by the variational method where the entropy of the $g$-on gas is employed. By means of an approximate solution to Haldane statistics, an elegant formula for the distribution function of the $g$-on gas is found. In Section 2.2, where the steepest descent method is applied to the partition function, an exact solution of the distribution function is obtained. In Section 2.3, the probability of the occupation numbers and average distribution functions are investigated by the statistical method. In Sections 2 and 3, fluctuations in the occupation number and thermodynamics of the $g$-on gas are discussed respectively.

## 2 Derivation of the distribution function of the g-on gas system

### 2.1 Variational method

Let us consider an ideal $g$-on gas system which consists of $N$ particles. Let the system be in heat and particle baths

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that evolves towards equilibrium, that is, the total energy $E$ and the total number of $g$-ons $N$ fluctuate, but, these quantities are conserved on the average. The entropy of the ideal $g$-on system which is not in equilibrium has been obtained from the statistical weight and is given by [1],

$$S_g = \sum_{k=1}^{K} g_k \{ [1 + n_k (1-g)]$$
$$\times \ln[1 + n_k (1-g) - (1-gn_k) \ln [1 - gn_k] - n_k \ln n_k \} \tag{1}$$

where the Boltzmann constant is set to unity, $g_k$ is the degeneracy, $n_k$ is the occupation number, $K$ is the total number of states; $g$ is the parameter that characterizes the complete or partial action of the exclusion principle *i.e.* makes interpolation between quantum gases [2,3]. Following the variational method outlined in the reference [1], the following equation is obtained

$$[\ln 1 + n_k (1-g)]^{(1-g)} + (1-g) + \ln (1 - gn_k)^g$$
$$+ g - \ln n_k - 1 - \alpha - \beta \varepsilon_k = 0 \tag{2}$$

where $\alpha = -\frac{\mu}{T}, \beta = \frac{1}{T}, \varepsilon_k$ is is the energy of the $k$th state and $\mu$ is the chemical potential. After performing some algebra equation (2) becomes:

$$\frac{[1 + n_k (1-g)]^{1-g} (1 - gn_k)^g}{n_k} = e^{x_k} \tag{3}$$

where

$$x_k = \beta (\varepsilon_k - \mu). \tag{4}$$

Equation (3) is known as Haldane statistics [4].

Let us write equation (3) in a more compact form

$$\frac{1}{y} (1-y)^g = e^x \tag{5}$$

where $y$ is defined as

$$y = \frac{1}{\frac{1}{n_k} + 1 - g}. \tag{6}$$

In order to get a real $x$ on the right hand side of equation (5), $y \leq 1$. This inequality imposes a condition

$$gn_k \leq 1. \tag{7}$$

Note that for $g = 1$ inequality (7) gives $n_k \leq 1$, for $g = 0$; $n_k$ can take any values and for the intermediate value $n_k$ is restricted as $n_k \leq \frac{1}{g}$. Therefore, inequality (7) could be interpreted as the generalization of the *Pauli Exclusion principle*.

Since it is difficult to solve equation (5), an approximation is needed for $y$. Expanding the factor $(1 - y)^g$ in a Taylor series about $y = 0$:

$$(1 - y)^g \approx 1 - gy + O\left(y^2\right). \tag{8}$$

Thus, equation (5) becomes

$$n_k = \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 2g - 1}. \tag{9}$$

The last approximation is a linear approximation to equation (3) and therefore does not satisfy inequality (7), if $g$ is not close enough to 0 or 1. Interesting enough, equation (9) turns out to be the Maxwell distribution when $g = \frac{1}{2}$ [1].

## 2.2 Steepest descent method

In order to get another solution of the distribution function $n_k$ we are willing to calculate it using the steepest descent optimization method [5]. Two of the authors of the present study FB and DD recently introduced a partition function $\mathcal{Z}_g$ for a grandcanonical ensemble of a $g$-on gas [1]. Let us write it for the extensive systems as

$$\mathcal{Z}_g = \prod_{k=1}^{K} \left\{ \sum_{k=0}^{1} e^{-x_k} + (1 - g) \sum_{k=2}^{\infty} e^{-x_k} \right\} \tag{10}$$

where $x_k$ is given by equation (4). The given partition function leads to the following partition function for the canonical ensemble

$$Z_g = \prod_{k=1}^{K} (1 + y_k) + (1 - g)\left(y_k^2 + y_k^3 + ...\right) \tag{11}$$

where $y_k = e^{-\beta \varepsilon_k}$ and $k$ denoting the quantum state $\varepsilon_k$. Here $g$ is a parameter that unifies the partition functions of boson and fermion systems under $g$-on gas system. For $g = 0$ the partition function in equation (11) becomes the partition function of a boson system, for $g = 1$ it becomes that of a fermion system. Let us find the distribution function of a $g$-on system using this partition function and the steepest descent method.

In order to use the steepest descent method one has to express the partition function as a complex integral around a closed contour. For this purpose let us define the function

$$f_g(z) = \prod_{k=1}^{K} \left\{ (1 + zy_k) + (1 - g)\left(z^2 y_k^2 + z^3 y_k^3 + ...\right) \right\}. \tag{12}$$

Using this function, the partition function given in equation (11) can be written as

$$Z_g = \frac{1}{2\pi i} \oint_C \frac{f_g(z)}{z^{N+1}} dz \tag{13}$$

or

$$Z_g = \frac{1}{2\pi i} \oint_C \prod_{k=1}^{K} \left[ \left( \frac{1 + zy_k}{z^{N+1}} + \frac{(1 - g) z^2 y_k^2 \left(1 + zy_k + z^2 y_k^2 + ...\right)}{z^{N+1}} \right) \right] dz. \tag{14}$$

This integral (or partition function) can be approximated by using the method of steepest descent. In order to see this let us write

$$e^{h(z)} = \frac{f_g(z)}{z^{N+1}} \tag{15}$$

where $f_g(z)$ is given by equation (12). Using this definition one can rewrite $h(z)$ as

$$h(z) = \ln \prod_{k=1}^{K} \left[ \left( \frac{1 + zx_k}{z^{N+1}} + \frac{(1 - g) z^2 y_k^2 \left(1 + zy_k + z^2 y_k^2 + ...\right)}{z^{N+1}} \right) \right] \tag{16}$$

or as

$$h(z) = \sum_{k=1}^{\infty} \left\{ \ln \left[ (1 + zy_k) + (1 - g) z^2 y_k^2 \left(1 + zy_k + z^2 y_k^2 + ...\right) \right] \right\} - \ln z^{-N+1}. \tag{17}$$

The function $h(z)$ has a minimum at $z_o$ and this minimum is very steep because the function $f_g(z)$ consists of infinite number of multiplication and $N + 1$ is very large. Thus by using the approximation formula of the steepest descent method, one gets an expression for the partition function of a $g$-on gas as [6]

$$Z_g \cong \left[ z_o^{-N+1} f_g(z_0) 2\pi f''(z_0) \right]^{1/2}. \tag{18}$$

In order to find the distribution function one applies the steepest descent condition to $h(z)$, that is

$$h'(z_0) = -\frac{N+1}{z_0} +$$
$$\sum_{k=0}^{\infty} \frac{y_k + (1-g)\left(2z_0 y_k^2 + 3z_0^2 y_k^3 + ...\right)}{1 + z_0 y_k + (1-g) z_0^2 y_k^2 \left(1 + z_0 y_k + z_0^2 y_k^2 + ...\right)} = 0. \tag{19}$$

Since $N \gg 1$, one can write $N + 1 \approx N$. Using this approximation and arranging equation (18) one obtains

$$N \approx \sum_{k=0}^{\infty} \frac{z_0 y_k + (1-g)\left(2z_0^2 y_k^2 + 3z_0^3 y_k^3 + ...\right)}{1 + z_0 y_k + (1-g) z_0^2 y_k^2 \left(1 + z_0 y_k + z_0^2 y_k^2 + ...\right)}. \tag{20}$$

The factor $\frac{1}{z_0}$ can be interpreted as fugacity $e^{-\beta\mu}$ [6]. For the sake of simplicity let us define a new variable as

$$a_k = e^{-\beta(\varepsilon_k - \mu)} = e^{-x_k} = z_0 y_k. \tag{21}$$

Using the series

$$1 + a_k + a_k^2 + a_k^3 + ... = (1 - a_k)^{-1} \tag{22}$$

$$1 + 2a_k + 3a_k^2 + ... = (1 - a_k)^{-2} \tag{23}$$

where $a_k < 1$. By means of these series and the definition (21) one can write equation (20) as

$$N = \sum_{k=1}^{\infty} \frac{a_k}{a_k + \frac{ga_k^2 - 2a_k + 1}{1 - 2ga_k + ga_k^2}} = \sum_{k=1}^{\infty} \frac{a_k\left(1 - 2ga_k + ga_k^2\right)}{\left(1 - a_k\right)\left(1 - ga_k^2\right)}. \tag{24}$$

or

$$N = \sum_{k=1}^{\infty} \frac{e^{-x_k}\left(1 - 2ge^{-x_k} + ge^{-2x_k}\right)}{\left(1 - e^{-x_k}\right)\left(1 - ge^{-2x_k}\right)}. \tag{25}$$

Since the total number of particles of a system is given as

$$N = \sum_{k=1}^{\infty} \overline{n_k} \tag{26}$$

the terms in equation (25) give the number of particles which occupy the $k$th state. Hence for the distribution of $g$-on particles it is possible to write

$$\overline{n_k} = \frac{e^{-x_k}\left(1 - 2ge^{-x_k} + ge^{-2x_k}\right)}{\left(1 - e^{-x_k}\right)\left(1 - ge^{-2x_k}\right)}. \tag{27}$$

Equation (27) which unifies the distribution of the quantum gases recovers the well known Bose and Fermi distributions for $g = 0$ and $g = 1$ respectively.

In order to find a more elegant result which could be used more easily in the calculations the term in equation (24) can be written as

$$\frac{ga_k^2 - 2a_k + 1}{1 - 2ga_k + ga_k^2} \approx 1 + (2g - 2)a_k. \tag{28}$$

By taking into account equation (28), equation (27) reads,

$$\overline{n_k} = \frac{a_k}{1 + a_k\left(2g - 1\right)} \tag{29}$$

which is valid near $g = 1$ and $g = 0$. Let us write equation (29) explicitly:

$$\overline{n_k} = \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 2g - 1}. \tag{30}$$

It is observed that equation (30) is same as equation (9) which is obtained through an approximation to Haldane formula.

## 2.3 Statistical method

The partition function of $g$-ons which is given by equation (10) can be used to find the probability of the $k$th state being occupied by $n_k$ $g$-ons. If the state $k$ with an energy $\varepsilon_k$ is occupied by $n_k$ $g$-ons then the other states are occupied by $N$-$n_k$ $g$-ons, where $N$ is the total particle number. Therefore the ratio of the partition function for the states except $k$th state times $e^{-n_k x_k}$ to the partition function gives us the probability of finding $n_k$ particles in the $k$th state. In the thermodynamic limit *i.e.* as $N \to \infty$ this probability can be written as

$$P_g^k(n_k) = \frac{e^{-n_k x_k} \prod_{\substack{i=1 \\ i \neq k}} \left\{ \sum_{n_i=0}^{1} e^{-n_i x_i} + (1-g)\sum_{n_i=2}^{\infty} e^{-n_i x_i} \right\}}{\prod_{i=1} \left\{ \sum_{n_i=0}^{1} e^{-n_i x_i} + (1-g)\sum_{n_k=2}^{\infty} e^{-n_i x_i} \right\}} \tag{31}$$

or writing in a simplified form:

$$P_g^k(n_k) = \frac{e^{-n_k x_k}}{\sum_{n_k=0}^{1} e^{-n_k x_i} + (1-g)\sum_{n_k=2}^{\infty} e^{-n_k x_i}}. \tag{32}$$

After performing the summation equation (32) turns out to be

$$P_g^k(n_k) = \frac{e^{-n_k x_k}\left(1 - ge^{-x_k}\right)}{1 - ge^{-2x_k}}. \tag{33}$$

It could be mentioned that equation (33) is a unified form of the probabilities of occupation number of the quantum gases simply bosons and fermions. One may write the distribution functions by means of equation (33) for $g$-ons as

$$\overline{n_k} = \sum_{n_k=0}^{1} n_k P_g^k(n_k) + (1-g)\sum_{n_k=2}^{\infty} n_k P_g^k(n_k). \tag{34}$$

After performing the summations it reads

$$\overline{n_k} = \frac{e^{-x_k}\left(1 - e^{-x_k}\right)}{1 - ge^{-2x_k}} \left[ g + \frac{1-g}{\left(1 - e^{-x_k}\right)^2} \right] =$$
$$\frac{e^{-x_k}\left(ge^{-2x_k} - 2ge^{-x_k} + 1\right)}{\left(1 - ge^{-2x_k}\right)\left(1 - e^{-x_k}\right)} \tag{35}$$

or more explicitly:

$$\overline{n_k} = \frac{e^{-\beta(\varepsilon_k-\mu)}\left(1 - 2ge^{-\beta(\varepsilon_k-\mu)} + ge^{-2\beta(\varepsilon_k-\mu)}\right)}{\left(1 - e^{-\beta(\varepsilon_k-\mu)}\right)\left(1 - ge^{-2\beta(\varepsilon_k-\mu)}\right)} \quad (36)$$

which is same as equation (27) as expected.

## 3 Fluctuations in the occupation number

In the preceding Section we have calculated the probability $P_g^k(n_k)$ of occupying a $g$-on of the $k$th state in the Fock space and the distribution function $\overline{n_k}$.

The deviations of mean values in the distribution function are given by the standard deviations of distributions,

$$\sigma^2 = \overline{n_k^2} - \overline{n_k}^2. \quad (37)$$

Therefore at first $\overline{n_k^2}$ has to be calculated. For this purpose let us write the following equation

$$\overline{n_k^2} = \sum_{n_k=0}^{1} n_k^2 P_g^k(n_k) + (1-g)\sum_{n_k=2}^{\infty} n_k^2 P_g^k(n_k). \quad (38)$$

Substituting the expression for $P_g^k(n_k)$ into equation (38) and performing the summations one finds

$$\overline{n_k^2} = \frac{e^{-x_k}\left(1 - e^{-x_k}\right)}{1 - ge^{-2x_k}}\left[g + (1-g)\frac{1 + e^{-x_k}}{\left(1 - e^{-x_k}\right)^3}\right]. \quad (39)$$

By means of equations (35), (37) and (38) one can calculate the variance as

$$\sigma^2 = \frac{e^{-x_k}(g^2e^{-4x_k} - 4ge^{-3x_k} + 6ge^{-2x_k} - 4ge^{-x_k} + 1)}{\left(1 - e^{-x_k}\right)^2\left(1 - ge^{-2x_k}\right)^2}. \quad (40)$$

and the ratio $\frac{\sigma^2}{\overline{n_k}^2}$ as

$$\frac{\sigma^2}{\overline{n_k}^2} = \frac{e^{x_k}(g^2e^{-4x_k} - 4ge^{-3x_k} + 6ge^{-2x_k} - 4ge^{-x_k} + 1)}{\left(ge^{-2x_k} - 2ge^{-x_k} + 1\right)^2} \quad (41)$$

which reduces to the standard results.

## 4 Thermodynamics of the g-on gas

In this Section, as a concrete application of the $g$-on statistics we would like to investigate the thermodynamics of an indistinguishable $g$-on gas. The ideal $g$-on gas is a model system, where the influence of quantum statistical effects can be very well studied. Our aim is to calculate the thermodynamical potential of the $g$-on gas $\Phi_g(T,V,\mu)$.

$$\Phi_g(T,V,\mu) = -k_BT\ln\mathcal{Z}_g(T,V,z) \quad (42)$$

where the summation is done over all of the quantum states. By taking into account equation (10) one derives approximately,

$$\Phi_g(T,V,\mu) = -k_BT\sum_{k=1}^{K}\ln\left[1 + \frac{1}{\omega_k}\right] \quad (43)$$

where

$$\omega_k \cong \frac{1}{e^{-x_k} + g - 1} \quad (44)$$

Since it is difficult to handle with the exact formula for $\overline{n_k}$ its approximation equation (10) or equation (30) is appropriate

$$\overline{n_k} = \frac{1}{\omega_k + g}. \quad (45)$$

Given the generalized thermodynamic potential

$$\Phi_g(T,V,\mu) = -p_gV \quad (46)$$

the pressure of the $g$-on gas is obtained as

$$p_gV = k_BT\sum_{k=1}^{K}\ln\left[1 + \frac{1}{\omega_k}\right] \quad (47)$$

while the total number of particles is found as

$$N_g = \sum_k\frac{1}{\omega_k + g}. \quad (48)$$

The state of the gas is determined by equations (47) and (48). But one can also calculate the energy of the gas which is given by

$$E_g = \sum_k\overline{n_k}\varepsilon_k = \sum_k\frac{\varepsilon_k}{\omega_k + g}. \quad (49)$$

The logarithm of the grandpartition function $Q_g(T,V,z)$ is given as

$$Q_g(T,V,z) = \ln\mathcal{Z}_g(T,V,z) = \sum_{k=1}^{K}\ln\left[1 + \frac{1}{\omega_k}\right]. \quad (50)$$

Let one of the $g$-on energies $\varepsilon_k$ be those of the energies of a free quantum mechanical particle in a box of volume $V$. The chemical potential or fugacity is not fixed, but the total number of $g$-ons are conserved on the average and $z$ has to be determined from equation (48). For a large volume the sum over all single $g$-on states can be given as the single $g$-on density states, where spin is not taken into account. Hence,

$$D(\varepsilon) = A\varepsilon^{1/2} \quad (51)$$

can be written where

$$A = \frac{2\pi V}{h^3}(2mh)^{3/2}. \quad (52)$$

Thus the summation in equation (50) becomes

$$Q_g(T,V,z) = \int_0^\infty\ln\left[1 + \frac{1}{\omega_k}\right]\varepsilon^{1/2}d\varepsilon \quad (53)$$

where equation (51) is taken into account. When the integration is performed by parts, $Q(T,V,z)$ is reduced to:

$$Q_g(T,V,z) = \frac{2}{3}A\beta\int_0^\infty\varepsilon^{3/2}nd\varepsilon. \quad (54)$$

After substituting $n$ from equation (45) and then rearranging

$$Q_g(T,V,z) = \frac{2}{3} A\beta z \int_0^\infty \frac{\varepsilon^{3/2}}{e^{\beta\varepsilon} - (1-2g)\,z} \mathrm{d}\varepsilon \qquad (55)$$

is obtained, where $z$ is the usual fugacity *i.e.* $z = e^{\beta\mu}$. The integral is calculated using the related formula on page 326 of reference [7]. Thus one concludes that for a $g$-on system

$$Q_g(T,V,z) = \frac{2}{3} A\beta z F_{5/2}(g,z) \qquad (56)$$

where the definition

$$F_n(g,z) = \frac{\Gamma(n)}{z(1-2g)\,\beta^n} \sum_{k=1}^{K} \frac{[z(1-2g)]^k}{k^n} \qquad (57)$$

is adopted in which $\Gamma(n)$ is the usual gamma function. Then the state of the $g$-on system is determined by the following equation:

$$p_g V = k_B T Q_g(T,V,z). \qquad (58)$$

Thus, the equation of state of fermions and bosons is unified.

In the thermodynamics limit, that is in the limit of infinite volume with particle density held fixed,particles in the ground state, having no contribution of the kinetic energy to the pressure, then, in terms of the thermal wavelength, the pressure of the Bose system becomes, as it is expected,

$$p_0 = \frac{kT}{\lambda^3} \sum_{k=1}^\infty \frac{z^k}{k^{5/2}} \qquad (59)$$

where $\Gamma\left(\frac{5}{2}\right) = 3\frac{\sqrt{\pi}}{4}$ is taken and $\lambda$ is the thermal wavelength [8].

In a similar manner, the thermodynamical properties of a Fermi system follows immediately from the logarithm of the grandpartition function of a $g$-on system, which is given by equation (43). In terms of the thermal wavelength, the pressure of the Fermi system is

$$p_1 = g_s \frac{k_B T}{\lambda^3} \sum_{k=1}^\infty \frac{(-1)^{k-1} z^k}{k^{5/2}} \qquad (60)$$

where $g_s$ is a factor due to spin [8].

For a large volume, in accordance with equation (49), the total number of $g$-ons could be written as

$$N_g(T,V,z) = Az \int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta\varepsilon} - (1-2g)\,z} \mathrm{d}\varepsilon \qquad (61)$$

assuming that single number of $g$-on states can be calculated in terms of integrals. Calculation of the integral in equation (61) using the related integrals in reference [7] on page 326 leads to

$$N_g(T,V,z) = Az F_{3/2}(g,z) \qquad (62)$$

where $F_{3/2}(g,z)$ is defined by equation (58) [8]. If $g = 0$ and $g = 1$ are substituted in equation (62) the formula for the total number of Bose system $N_0(T,V,z)$ and Fermi system $N_1(T,V,z)$ are respectively recovered .

For a Bose system the total number of particles $N_0(T,V,z)$ including $\varepsilon = 0$ energy states, is found to be:

$$N_0(T,V,z) = Az F_{3/2}(z) + \frac{z}{1-z} \qquad (63)$$

where the last term represents the contribution of the energy level $\varepsilon = 0$ to the mean particle number.

In a similar manner, for the total number of fermions one gets

$$N_1(T,V,z) = g_s \frac{V}{\lambda^3} \sum_{k=1}^\infty \frac{(-1)^{k-1} z^k}{k^{1/2}} \qquad (64)$$

where $\Gamma(3/2) = \sqrt{\pi/2}$ is taken. This is an expected standard result [8].

# 5 Conclusions

This study has been initiated from the point of view that bosons and fermions, which obey quantum statistics, could be regarded as $g$-ons which obey fractional statistics. In this paper, Haldane statistics is rederived by variational method where entropy of the $g$-on gas is involved. Due to the importance of the distribution function, by means of steepest descent and statistical methods, mean occupation number is derived where partition function of $g$-on gas is used. It is concluded that mean occupation number which is found by these methods lead to same expressions. Within this context the probabilities of occupation number of $g$-on gas is also calculated which unifies the probabilities of quantum gases.

The approximate solutions of the distribution function of a $g$-on gas system, which is found from the variational, steepest descent and statistical methods are developed to an elegant expression that recovers FD distribution for $g = 1$, and BE distribution for $g = 0$. It should be remarked that the partition function which is used in this study is appropriate for the $g$-on gas since it leads to the same approximate expression for the mean occupation number with Haldane statistics.

As an application of the mean occupation number, the thermodynamical formulation of the $g$-on gas is derived which gives the opportunity to unify the thermostatistical formulation of quantum gases, simply fermions and bosons.

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